$1/12+x/12+x^2/12+x^3/12+x^4/12+x^5/12+$ ....(13).

Let $A_n{}^1$, $A_n{}^2$, $A_n{}^3$, $A_n{}^4$, $S_n{}^1$, $S_n{}^2$, $S_n{}^3$, $S_n{}^4$ represent the nth terms, and the sum of n terms of the series (10), (11), (12), (13). Then,

$$A_n{}^1=-\tfrac{2}{3}(2x)^{n-1},\ A_n{}^2=\tfrac{2}{21}(\pm 2x)^{n-1},\ A_n{}^3=\tfrac{125}{84}(5x)^{n-1},\ A_n{}^4=\tfrac{1}{12}x^{n-1},$$

$$S_n{}^1=-\tfrac{2}{3}\left(\frac{2^nx^n-1}{2x-1}\right),\ S_n{}^2=\tfrac{2}{21}\left(\frac{\pm 2^nx^n-1}{-2x-1}\right),\ S_n{}^3=\tfrac{125}{84}\left(\frac{5^nx^n-1}{5x-1}\right),$$

$$S_n{}^4=\tfrac{1}{12}\left(\frac{x^n-1}{x-1}\right).$$

Let A_n, S_n, be the nth term and the sum of n terms of the original series.

$$\therefore\ A_n=\tfrac{1}{84}\{5^{n+2}+7(1-2^{n+2})\mp 2^{n+2}\}x^{n-1}.$$

$$S_n=\tfrac{1}{84}\left\{\frac{125(5^nx^n-1)}{5x-1}-\frac{56(2^nx^n-1)}{2x-1}+\frac{8(\pm 2^nx^n-1}{-2x-1}+\frac{7(x^n-1)}{x-1}\right\}.$$

The upper sign to be used when n is even. Now let $x=1$, $n=20$, and we will get the required results for the problem. $A_{20}=28383163779300$, the number the twentieth year ; $S_{20}=35478954491110$, the number in twenty years.

Also solved by *EDWARD R. ROBBINS*.

65. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A, B, and C bought unequal shares in 200 acres of land at the same price per acre, which they sold for \$286.90. A gained as much per cent. on his part as he had acres, B gained 5-8 as much per cent. on his part as A did, and C lost \$9.10 on the cost of his part; the total net gain was 43 9-20 per cent. How much land did each buy, and what did each receive per acre at the sale?

I. Solution by W. H. CARTER, Professor of Mathematics in Centenary College of Louisiana, Jackson, Louisiana.

Let x, y, and z be the number of acres bought by A, B, and C, respectively. $\therefore\ x+y+z=200$ ....(1).

Since the selling price is \$286.90 and the gain per cent. is 43.45, the cost is \$200. Let m=cost per acre ; then mx, my, and mz represent the cost of the shares of A, B, and C, respectively. $\therefore\ m(x+y+z)=200$. $\therefore\ m=1$. $\therefore$ the cost of the share of each=number of acres he bought.

x=A's gain per cent., and $5x/8$=B's gain per cent.

$\therefore\ x+x^2/100+y+5xy/800+z-\$9.10=\$286.90$.

$\therefore\ x^2/100+5xy/800=\$96$. $\therefore\ 8x^2+5xy=76800$.

$$\therefore\ y=\frac{76800-8x^2}{5x}=\frac{15360}{x}-\frac{8x}{5} \quad \text{..................................(2).}$$

If the number of acres bought by each is to be integral, then (1) and (2) are to be solved for *positive integral* values of x, y, and z. Since y is to be integral, x must be a factor 15360 and must be divisible by 5. $15360=5\times3\times2^{10}$. $\therefore$ the factors of 15360 which are divisible by 5, are 5, 10, 15, 20, 30, 40, 60, 80, 120, 160, etc. If x has any of these values less than 80, z will be negative; if x has any values greater than 80, y is negative. If $x=80$, $y=64$, and $z=56$. $\therefore$ 80, 64, and 56 are the shares of A, B, and C.

The amounts each received per acre at the sale are easily found to be \$1.80, \$1.50, and \$0.83$\frac{3}{4}$.

II. Solution by EDWARD R. ROBBINS, Master in Mathematics and Physics in the Lawrenceville School, Lawrenceville, New Jersey.

Let x, y, and $200-x-y$ represent the number of acres which A, B, and C bought, respectively. Then by the problem,

$$x+x^2/100+y+5xy/800+200-x-y-9.10=286.90.$$

This gives $8x^2+5xy=76{,}800$; or $y=(76800-8x^2)/5x$. Solving for positive integers in x, we have, when

$$\begin{array}{lllll} x= & 75, & 80, & 85, & 90, \\ y= & 107\frac{3}{7}, & 64, & 44\frac{12}{17}, & 26\frac{2}{3}, \end{array}$$

Accepting the integral values we obtain:
A's purchase consisted of 80 acres and sold for \$144;
B's purchase consisted of 64 acres and sold for \$96;
C's purchase consisted of 56 acres and sold for \$46.90.
Hence A received \$1$\frac{4}{5}$ per acre; B, \$1$\frac{1}{2}$; and C, \$$\frac{67}{80}$.

III. Solution by H. C. WILKES, Skull Run, West Virginia.

Since by the terms of the problem the price paid for the land was \$1 per acre, let $8x$, y, z be the number of acres bought, and the number of dollars paid, by A, B, and C, respectively.

Then $8x+y+z=200\ldots(1)$. $8x+64x^2/100+y+5xy/100+z=296\ldots(2)$.

Subtracting (1) from (2), and clearing, $64x^2+5xy=9600$. Factoring, $x(64x+5y)=10(960)$. Let $x=10$; then $5y=320$, and $y=64$.

$\therefore$ 80, 64, 56 are numbers satisfying the conditions. See solution of a similar problem on page 76 of Vol. II.

IV. Solution by A. M. HUGHLETT, A. M., Associate Principal and Professor of Mathematics in Randolph-Macon Academy, Bedford City, Virginia.

Let x, y, and z represent the shares of A, B, and C, respectively. $x+y+z=200\ldots\ldots(1)$. Since C lost \$9.10, he must have bought at least 9.10 acres. Therefore 190.90 is the maximum limit of $x+y$.

$$x+x^2/100+y+xy/160+z=296\ldots\ldots(2).$$

(1) in (2) gives $x^2/100+xy/160=96$. $\therefore$ $y=(76800-8x^2)/5x\ldots\ldots(3)$.

$\therefore\ 190.90 > x+(76800-8x^2)/5x$. $\therefore\ 190.90 > (76800-3x^2)/5x$.
As x decreases, $(76800-3x^2)/5x$ increases.

$\therefore$ the equation $(76800-3x^2)/5x=190.90$..........................(4)

gives the minimum limit of x.

$\therefore$ 66.48+ is the minimum limit of x.............................(5).

From (3), $y=(76800-8x^2)/5x$, we get, since y must have some value, $76800>8x^2$; hence $8x^2=76800$ gives maximum limit of x. $\therefore$ 97.97+ is the maximum limit of x. Hence, any values of x between 66.48+ and 97.97+ will satisfy the conditions of the problem. *Example:* Let $x=77\frac{1}{7}$. Then from (3) $y=75\frac{43}{63}$; $\therefore\ z=47\frac{11}{63}$.

$\therefore$ A received \$136.65$\frac{135}{441}$; B received \$112.17$\frac{103}{441}$. $\therefore$ C received \$38.07$\frac{203}{441}$; but he paid \$47.17$\frac{203}{441}$. $\therefore$ C lost \$9.10.

Also solved by *A. H. HOLMES*, *J. SCHEFFER*, and *G. B. M. ZERR*.

PROBLEMS.

72. Proposed by CHAS. C. CROSS, Laytonsville, Maryland.

Prove that $\dfrac{2\sqrt{2+\sqrt{3}}}{4\times\sqrt{6-\sqrt{2}}}=\sqrt{6}-\sqrt{2}+\sqrt{3}-2$, when reduced to its lowest terms.

73. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Find the worth of each of five persons, A, B, C, D, and E, knowing, 1st, that when A's worth is added to a times what B, C, D, and E are worth, it is equal to m; 2nd, when B's worth is added to b times what A, C, D, and E are worth, it is equal to n; 3rd, when C's worth is added to c times what A, B, D, and E are worth, it is equal to p; 4th, when D's worth is added to d times what A, B, C, and E are worth, it is equal to q; 5th, when E's worth is added to e times what A, B, C, and D are worth, it is equal to r.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

51. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the maximum ellipsoid that can be cut out of a given right conic frustum.